Optimal Measurements of Spin Direction

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The accuracy of a measurement of the spin direction of a spin-s particle is characterized for arbitrary half-integral s. The disturbance caused by the measurement is also characterized. The approach is based on that taken in several previous papers concerning joint measurements of position and momentum. As in those papers, a distinction is made between the errors of retrodiction and prediction. Retrodictive and predictive error relationships are derived. The POVM describing the outcome of a maximally accurate measurement process is investigated. It is shown that if the measurement is retrodictively optimal, then the distribution of measured values is given by the initial state SU(2) Q-function. If the measurement is predictively optimal, then the distribution of measured values is related to the final state SU(2) P-function. The general form of the unitary evolution operator producing an optimal measurement is characterized.

1. INTRODUCTION

In a recent series of papers (Appleby, 1998a–c, 1999a, b) we analyzed the concept of experimental accuracy as it applies to simultaneous measurements of position and momentum (Arthurs and Kelly, 1965; Stenholm, 1992; Peres, 1993; Raymer, 1994; Busch *et al.*, 1995; Schroeck, 1996; Leonhardt, 1997; and references cited therein). The purpose of this paper is to give a similar analysis for measurements of spin direction.

There have been a number of previous discussions of joint, imperfectly accurate measurements of two (noncommuting) components of spin (Prugovečki, 1977; Schroeck 1982; Busch, 1986, 1987, 1988; Martens and de Muynck, 1993; Kienzler, 1998). Measurements of spin direction—the kind of measurement considered in this paper—have been discussed by Busch and Schroeck (1989), Grabowski (1989), Peres (1993), and Busch *et al* (1995). In the following we extend the work of these authors by giving an analysis

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of the measurement errors and of the conditions for a measurement process to be optimal. In particular, we will show that a measurement is retrodictively optimal if and only if the distribution of measured values is given by the generalized Q-function (Radcliffe, 1971; Arecchi *et al.*, 1972; Lieb, 1973; Berezin, 1975; Perelomov, 1986, Várilly and Gracia-Bondía, 1989; Amiet and Cibils, 1991) which is defined in terms of SU(2) coherent states. This corresponds to an analogous property of joint measurements of position and momentum derived by Ali and Prugovečki (1977) and proved under less restrictive conditions in Appleby (1999a).

This result provides us with some further insight into the physical significance of the SU(2) Q-function, It also has a bearing on the problem of state reconstruction. Amiet and Weigert (1998, 1999a–c) and Weigert (1998, 1999) have recently shown how, by making measurements of a single spin component for sufficiently many differently oriented Stern–Gerlach apparatuses, one can calculate the corresponding values of the SU(2) Q-function, and thereby reconstruct the density matrix. The fact that a retrodictively optimal measurement of spin direction has the Q-function as its distribution of measured values suggests an alternative approach to the problem of state reconstruction: for it means that one can reconstruct the density matrix from the statistics of a single run of measurements performed on a single apparatus. The fact that measurements of spin direction whose outcome is described the Q-function have this property of informational completeness has been stressed by Busch and Schroeck (1989) (also see Busch, 1991; Busch *et al*, 1995; Schroeck, 199).

Retrodictively optimal joint measurements of position and momentum (Ali and Prugovečki, 1977; Appleby, 1999a) give rise to the ordinary Husimi or Q-function (Husimi, l940; Hillery *et al.*, 1984; Lee, 1995; and references cited therein), and so they also have the property of informational completeness (Busch, 1991; Busch *et al.*, 1995; Schroeck, 1996; Nakajima, 1999), at least in principle. However, the practical usefulness of this fact is somewhat restricted due to the amplification of statistical errors which occurs when one attempts to perform the reconstruction starting from real experimental data (Leonhardt and Paul, 1994a, b; Poyatos *et al.*, 1996; Leonhardt, 1997; Banaszek, 1999; Appleby, 1999c). No such difficulty arises in the case of measurements of spin direction due to the fact that the state space is finite dimensional.

We now outline the approach taken in the remainder of this paper. We consider a system consisting of a single spin, with angular momentum operator $\hat{\mathbf{S}}$ satisfying the usual commutation relations $[\hat{S}_a, \hat{S}_b] = i \sum_{c=1}^{3} \epsilon_{abc} \hat{S}_c$ (with units chosen such that $\hbar = 1$). We take it that $\hat{S}^2 = s(s + 1)$ for some arbitrary, but fixed half-integer s.

The components of $\hat{\mathbf{S}}$ are noncommuting, so they cannot all be simultaneously measured with perfect precision. However, they can all be measured with a less than perfect degree of accuracy. In order to do so one can use the same kind of procedure which is employed in the Arthurs–Kelly process (Arthurs and Kelly, 1965; Stenholm, 1992; Peres, 1993; Busch *et al.*, 1995; Schroeck, 1996; Leonhardt, 1997; Appleby, 1998b): that is, one can couple the noncommuting observables of interest—the components of $\hat{\mathbf{S}}$—to another set of "pointer" or "meter" observables which do commute and whose values may therefore be simultaneously determined with arbitrary precision.

The question we then have to decide is how to choose the pointer observables. The observables to be measured satisfy the constraint $\hat{S}^2 = s(s+1)$, where s is fixed. Consequently, one might take the view that the magnitude of the spin vector is already known and that all that needs to be measured is its direction. This suggests that the pointer observables should be taken to be the (commuting) components of a unit vector $\hat{\mathbf{n}}$ satisfying the constraint $\hat{n}^2 = 1$. The direction of $\hat{\mathbf{n}}$ measures the direction of $\hat{\mathbf{S}}$. We will refer to this as a type 1 measurement. Such measurements are discussed in Sections 2–7.

There is another possibility: one could take the pointer observables to be the three *independent*, commuting components of a vector $\hat{\boldsymbol{\mu}}$, no constraint being placed on the squared modulus $\hat{\mu}^2$. The value of $\hat{S}_1$ (respectively $\hat{S}_2$, $\hat{S}_3$) is measured by $\hat{\mu}_1$ (respectively $\hat{\mu}_2$, $\hat{\mu}_3$). We will refer to this as a type 2 measurement. Such measurements are discussed in Section 8.

We begin our analysis in Section 2, by characterizing the POVM (positive operator valued measure) describing the outcome of an arbitrary type 1 measurement process.

In Section 3 we characterize the accuracy of and disturbance caused by a type 1 measurement process. Our definitions are based on those given in Appleby (1998a, c) for simultaneous measurements of position and momentum. In particular, we are led to make a distinction between two different kinds of accuracy, which we refer to as retrodictive and predictive.

After giving in Section 4 a brief summary of the relevant features of the theory of SU(2) coherent states, we go on in Section 5 to describe retrodictively optimal type 1 measurements. We establish a bound on the retrodictive accuracy. We define a retrodictively optimal measurement to be a measurement which (1) achieves the maximum possible degree of retrodictive accuracy and (2) is isotropic (in a sense to be explained). We then show that the necessary and sufficient condition for the measurement to be retrodictively optimal is that the distribution of measured values be given by the initial state SU(2) Q-function.

In Section 6 we establish a bound on the predictive accuracy of a type 1 measurement. We derive a necessary and sufficient condition for this bound

to be achieved, in which case we say that the measurement is predictively optimal. We show that the distribution of measured values is then related to the final state SU(2) *P*-function.

In Section 7 we consider completely optimal type 1 measurement process, i.e., processes that are both retrodictively and predictively optimal. We give the general form of the unitary evolution operator describing such a process.

Finally, in Section 8, we consider type 2 measurements. We define the retrodictive and predictive errors of such measurements and establish bounds which the errors must satisfy. We then show that in the limit as a type 2 measurement tends to optimality (retrodictive or predictive), it more and more nearly approaches an optimal type 1 measurement (with the replacement $s^{-1}\hat{\boldsymbol{\mu}} \to \hat{\mathbf{n}}$). It follows that, insofar as the aim is to maximize the measurement accuracy, type 2 measurements have no advantages.

2. TYPE 1 MEASUREMENTS: POVM

The purpose of this section is to characterize the positive operator valued measure (POVM; Kraus, 1983; Peres, 1993, 1999; Busch *et al.*, 1995; Schroeck, 1996; and references cited therein) describing the outcome of an arbitrary type 1 measurement.

We take a type 1 measurement to consist of a process in which the system with $(2s + 1)$-dimensional state space $\mathcal{H}_{\text{sy}}$ is coupled to a measuring apparatus, with state space $\mathcal{H}_{\text{ap}}$. The interaction commences at a time $t = t_i$ when system + apparatus is in the product state $|\psi \otimes \chi_{\text{ap}}\rangle$, where $|\psi\rangle \in \mathcal{H}_{\text{sy}}$ is the initial state of the system and $|\chi_{\text{ap}}\rangle \in \mathcal{H}_{\text{ap}}$ is the initial state of the apparatus. It ends after a finite time interval at $t = t_f$ when system + apparatus is in the state $\hat{U}|\psi \otimes \chi_{\text{ap}}\rangle$, where $\hat{U}$ is the unitary evolution operator describing the measurement interaction.

It should be stressed that this description is quite general. In particular, we are not making an impulsive approximation. Nor are we assuming that the interaction Hamiltonian is large in comparison with the Hamiltonians describing the system and apparatus separately. The only substantive assumption is the statement that system + apparatus is initially in a product state (so that they are initially uncorrelated).

It should be noted that $|\psi\rangle$ is arbitrary, since the system might initially be in any state $\in \mathcal{H}_{\text{sy}}$. On the other hand, $|\chi_{\text{ap}}\rangle$ is fixed, since we assume that initially the apparatus is always in the same "zeroed" or "ready" state.

As explained in Section 1, we take it that the result of the measurement is specified by the recorded values of three commuting pointer observables $\hat{\mathbf{n}} = (\hat{n}_1, \hat{n}_2, \hat{n}_3)$, satisfying the constraint $\sum_{r=1}^{3} \hat{n}_r^2 = 1$ (so that there are only two pointer degrees of freedom). However, a measuring instrument does not

usually consist of some pointers and nothing else. We therefore allow for the existence of N additional apparatus observables $\hat{\xi} = (\hat{\xi}_1, \ldots, \hat{\xi}_N)$ which, together with the components of $\hat{\mathbf{n}}$, constitute a complete commuting set. The eigenkets $|\mathbf{n}, \xi\rangle$ thus provide an orthonormal basis for $\mathcal{H}_{ap}$.

The operator $\hat{U}$ specifies the final state of system $+$ apparatus given *any* initial state $\in \mathcal{H}_{sy} \otimes \mathcal{H}_{ap}$. However, we are only interested in initial states of the very special form $|\psi \otimes \chi_{ap}\rangle$, where $|\chi_{ap}\rangle$ is fixed. In other words, the operator $\hat{U}$ provides us with much more information than we actually need. It turns out that all the quantities which are relevant to the argument of this paper can be expressed in terms of the operator $\hat{T}(\mathbf{n}, \xi)$, defined by (Kraus, 1983; Busch *et al.*, 1995; Schroeck, 1996; Peres, 1999)

$$\hat{T}(\mathbf{n}, \xi) = \sum_{m_1, m_2 = -s}^{s} (\langle m_1| \otimes \langle \mathbf{n}, \xi|)\hat{U}(|m_2\rangle \otimes |\chi_{ap}\rangle)|m_1\rangle\langle m_2| \qquad (1)$$

where $|m\rangle$ denotes the eigenket of $\hat{S}_3$ with eigenvalue m (in units such that $\hbar = 1$). The operator $\hat{T}(\mathbf{n}, \xi)$ is more convenient to work with because, unlike $\hat{U}$, it only acts on the system state space $\mathcal{H}_{sy}$.

The significance of the operator $\hat{T}(\mathbf{n}, \xi)$ is that it describes the change in the state of the system which is caused by the measurement process (Kraus, 1983; Busch *et al.*, 1995; Schroeck, 1996; Peres, 1999) (i.e., it describes the operation induced by the measurement). In fact, suppose that the measurement is nonselective (meaning that the final value of $\mathbf{n}$ is not recorded, so that there is no "collapse"), and let $\hat{\rho}_f$ be the reduced density matrix describing the final state of the system. It is then readily verified that

$$\hat{\rho}_f = \int d\mathbf{n} \, d\xi \, \hat{T}(\mathbf{n}, \xi)|\psi\rangle\langle\psi|\hat{T}^\dagger(\mathbf{n}, \xi) \qquad (2)$$

where $d\mathbf{n}$ denotes the usual measure on the unit 2-sphere: in terms of spherical polars $d\mathbf{n} = \sin\theta \, d\theta \, d\phi$.

Let $\rho_{val}(\mathbf{n})$ be the probability density function describing the distribution of measured values:

$$\rho_{val}(\mathbf{n}) = \sum_{m=-l}^{l} \int d\xi \, |(\langle m| \otimes \langle \mathbf{n}, \xi|)\hat{U}(|\psi\rangle \otimes |\chi_{ap}\rangle)|^2 \qquad (3)$$

$\rho_{val}(\mathbf{n})$ can also be expressed in terms of the operators $\hat{T}(\mathbf{n}, \xi)$. In fact, define (Kraus, 1983; Busch *et al.*, 1995; Schroeck, 1996; Peres, 1999)

$$\hat{E}(\mathbf{n}) = \int d\xi \, \hat{T}^\dagger(\mathbf{n}, \xi)\hat{T}(\mathbf{n}, \xi) \qquad (4)$$

Then

$$\rho_{\text{val}}(\mathbf{n}) = \langle \psi | \hat{E}(\mathbf{n}) | \psi \rangle \tag{5}$$

We see from this that $\hat{E}(\mathbf{n}) \, d\mathbf{n}$ is the POVM describing the measurement outcome. In particular,

$$\hat{E}(\mathbf{n}) \geq 0$$

for all $\mathbf{n}$ and

$$\int d\mathbf{n} \, \hat{E}(\mathbf{n}) = 1 \tag{6}$$

Until now we have been assuming that the system is initially in a pure state. If the system is initially in the mixed state with density matrix $\hat{\rho}_i$ we have, in place of Eqs. (2) and (5),

$$\hat{\rho}_f = \int d\mathbf{n} \, d\xi \, \hat{T}(\mathbf{n}, \xi) \hat{\rho}_i \hat{T}^\dagger(\mathbf{n}, \xi) \tag{7}$$

and

$$\rho_{\text{val}}(\mathbf{n}) = \text{Tr}(\hat{E}(\mathbf{n}) \hat{\rho}_i) \tag{8}$$

Equation (7) gives the final state reduced density matrix for the system in the case when the measurement is nonselective, so that the pointer position is not recorded. Suppose, on the other hand, that the final pointer position is recorded to be in the subset $\mathcal{R}$ of the unit 2-sphere. Then $\hat{\rho}_f$ is given by

$$\hat{\rho}_f = \frac{1}{p_{\mathcal{R}}} \int_{\mathcal{R}} d\mathbf{n} \int d\xi \, \hat{T}(\mathbf{n}, \xi) \hat{\rho}_i \hat{T}^\dagger(\mathbf{n}, \xi) \tag{9}$$

where $p_{\mathcal{R}}$ is the probability of finding $\mathbf{n} \in \mathcal{R}$:

$$p_{\mathcal{R}} = \int_{\mathcal{R}} d\mathbf{n} \, \rho_{\text{val}}(\mathbf{n})$$

3. TYPE 1 MEASUREMENTS: ACCURACY AND DISTURBANCE

The purpose of this paper is to establish the form of the operators $\hat{T}(\mathbf{n}, \xi)$ and $\hat{E}(\mathbf{n})$ when the measurement is optimal. In order to give a precise definition of what "optimal" means in this context, we first need to define a concept of measurement accuracy, which is the problem addressed in this section. We also discuss how to quantify the degree to which the system is disturbed by the measurement process.

The approach we take is based on the approach taken in Appleby (1998a, c) to the problem of defining the accuracy of and disturbance caused by a simultaneous measurement of position and momentum. We thus work in terms of the Heisenberg picture.

Let $\hat{\mathbf{S}}_i = \hat{\mathbf{S}}$ and $\hat{\mathbf{n}}_i = \hat{\mathbf{n}}$ be the initial values of the Heisenberg spin and pointer observables at the time t_i, when the measurement interaction begins; and let $\hat{\mathbf{S}}_f = \hat{U}^\dagger \mathbf{S} \hat{U}$ and $\hat{\mathbf{n}}_f = \hat{U}^\dagger \hat{\mathbf{n}} \hat{U}$ be the final values of these observables at the time t_f, when the measurement interaction ends. Let $\mathscr{S}_{sy} \subset \mathscr{H}_{sy}$ be the unit sphere in the system state space. We then define the retrodictive fidelity η_i by

$$\eta_i = \inf_{|\psi\rangle \in \mathscr{S}_{sy}} (\langle \psi \otimes \chi_{ap}| \tfrac{1}{2}(\hat{\mathbf{n}}_f \cdot \hat{\mathbf{S}}_i + \hat{\mathbf{S}}_i \cdot \hat{\mathbf{n}}_f)|\psi \otimes \chi_{ap}\rangle) \qquad (10)$$

and the predictive fidelity η_f by

$$\eta_f = \inf_{|\psi\rangle \in \mathscr{S}_{sy}} (\langle \psi \otimes \chi_{ap}| \tfrac{1}{2}(\hat{\mathbf{n}}_f \cdot \hat{\mathbf{S}}_f + \hat{\mathbf{S}}_f \cdot \hat{\mathbf{n}}_f)| \psi \otimes \chi_{ap}\rangle)$$

$$= \inf_{|\psi\rangle \in \mathscr{S}_{sy}} (\langle \psi \otimes \chi_{ap}|\hat{\mathbf{n}}_f \cdot \hat{S}_f|\psi \otimes \chi_{ap}\rangle) \qquad (11)$$

(where we have used the fact that the components of $\hat{\mathbf{n}}_f$ and $\hat{\mathbf{S}}_f$ commute). It should be noted that the concept of fidelity employed here is somewhat different from the concept of fidelity employed in discussions of cloning and state estimation (η_i and η_f are defined in terms of scalar products of observables, rather than scalar products of states).

We also define the quantity η_d by

$$\eta_d = \inf_{|\psi\rangle \in \mathscr{S}_{sy}} (\langle \psi \otimes \chi_{ap}| \tfrac{1}{2}(\hat{\mathbf{S}}_f \cdot \hat{\mathbf{S}}_i + \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_f)|\psi \otimes \chi_{ap}\rangle) \qquad (12)$$

The intuitive basis for these definitions is most easily appreciated if one thinks, temporarily, in classical terms. If interpreted classically, η_i would represent the minimum expected degree of alignment between the final pointer direction and the initial direction of the spin vector. In other words, it would quantify the retrodictive accuracy of the measurement. On the other hand, η_f would represent the minimum expected degree of alignment between the final pointer direction and the final direction of the spin vector: it would therefore provide a quantitative indication of the predictive accuracy. Lastly, η_d would quantify the extent to which the measurement disturbs the system by changing the direction of the spin vector.

Of course, $\hat{\mathbf{n}}_f$, $\hat{\mathbf{S}}_i$, $\hat{\mathbf{S}}_f$ are in fact quantum mechanical observables, and so the physical interpretation of η_i, η_f, and η_d needs to be justified much more carefully. Rather than proceeding directly, it will be convenient first to relate these quantities to an alternative characterization of the measurement

accuracy and disturbance. This will allow us to appeal to the arguments given in Appleby (1998a, c) to justify our earlier characterization of the accuracy of and disturbance caused by a simultaneous measurement of position and momentum. It will also be helpful in Section 8 when we compare type 1 and type 2 measurements.

In a type 1 measurement, the result of the measurement is a direction, represented by the unit vector $\mathbf{n}$. However, one could extract from this information estimates of the initial and final values of the spin vector itself by multiplying $\mathbf{n}$ by suitable constants: say $\zeta_i \mathbf{n}$ as an estimate for $\mathbf{S}_i$, and $\zeta_f \mathbf{n}$ as an estimate for $\mathbf{S}_f$. The question then arises: what are the best choices for these constants?

To answer this question, consider the quantities

$$\sup_{|\psi\rangle \in \mathscr{S}_{sy}} (\langle \psi \otimes \chi_{ap}||\zeta_i \hat{\mathbf{n}}_f - \hat{\mathbf{S}}_i|^2|\psi \otimes \chi_{ap}\rangle) = \zeta_i^2 - 2\zeta_i \eta_i + s(s+1)$$

and

$$\sup_{|\psi\rangle \in \mathscr{S}_{sy}} (\langle \psi \otimes \chi_{ap}||\zeta_f \hat{\mathbf{n}}_f - \hat{\mathbf{S}}_f|^2|\psi \otimes \chi_{ap}\rangle) = \zeta_f^2 - 2\zeta_f \eta_f + s(s+1)$$

These expressions are minimized if we choose $\xi_i = \eta_i$, $\zeta_f = \eta_f$. We accordingly define the maximal rms error of retrodiction

$$\Delta_{ei}S = \left(\sup_{|\psi\rangle \in \mathscr{S}_{sy}} (\langle \psi \otimes \chi_{ap}||\eta_i \hat{\mathbf{n}}_f - \hat{\mathbf{S}}_i|^2|\psi \otimes \chi_{ap}\rangle) \right)^{1/2} = (s + s^2 - \eta_i^2)^{1/2}$$

(13)

and the maximal rms error of prediction

$$\Delta_{ef}S = \left(\sup_{|\psi\rangle \in \mathscr{S}_{sy}} (\langle \psi \otimes \chi_{ap}||\eta_f \hat{\mathbf{n}}_f - \hat{\mathbf{S}}_f|^2|\psi \otimes \chi_{ap}\rangle) \right)^{1/2} = (s + s^2 - \eta_f^2)^{1/2}$$

(14)

We also define the maximal rms disturbance by

$$\Delta_d S = \left(\sup_{|\psi\rangle \in \mathscr{S}_{sy}} (\langle \psi \otimes \chi_{ap}||\hat{\mathbf{S}}_f - \hat{\mathbf{S}}_i|^2|\psi \otimes \chi_{ap}\rangle) \right)^{1/2} = \sqrt{2}(s + s^2 - \eta_d)^{1/2}$$

(15)

Comparing these expressions with those given in Appleby (1998a, c), it can be seen that $\Delta_{ei}S$ plays the same role in relation to the kind of measurement here considered as do the quantities $\Delta_{ei}x$, $\Delta_{ei}p$ in relation to joint measurements of position and momentum, that $\Delta_{ef}S$ is the analogue of $\Delta_{ef}x$, $\Delta_{ef}p$, and that

$\Delta_d S$ is the analogue of $\Delta_d x$, $\Delta_d p$. A suitably modified version of the argument given in Section 5 of Appleby (1998a) may then be used to show that $\Delta_{ei}S$ (and therefore η_i) describes the retrodictive accuracy of the measurement, that $\Delta_{ef}S$ (and therefore η_f) describes the predictive accuracy, and that $\Delta_d S$ (and therefore η_d) describes the degree of disturbance caused by the measurement.

Finally, we note that the quantities η_i, η_f, and η_d can be expressed in terms of the operators $\hat{T}(\mathbf{n}, \xi)$ and $\hat{E}(\mathbf{n})$ defined earlier. In fact, comparing Eqs. (1) and (4) with Eqs. (10)–(12), one finds

$$\eta_f = \inf_{|\psi\rangle \in \mathscr{S}_{sy}} \left(\int d\mathbf{n} \, \langle\psi|\tfrac{1}{2}(\hat{E}(\mathbf{n})\mathbf{n} \cdot \hat{\mathbf{S}} + \mathbf{n} \cdot \hat{\mathbf{S}}\hat{E}(\mathbf{n}))|\psi\rangle \right) \tag{16}$$

$$\eta_f = \inf_{|\psi\rangle \in \mathscr{S}_{sy}} \left(\int d\mathbf{n} \, d\xi \langle\psi|\hat{T}^{\dagger}(\mathbf{n}, \xi)\mathbf{n} \cdot \hat{\mathbf{S}}\hat{T}(\mathbf{n}, \xi)|\psi\rangle \right) \tag{17}$$

and

$$\eta_d = \inf_{|\psi\rangle \in \mathscr{S}_{sy}} \left(\int d\mathbf{n} \, d\xi \sum_{a=1}^{3} \langle\psi|\tfrac{1}{2}(\hat{T}^{\dagger}(\mathbf{n}, \xi)\hat{S}_a\hat{T}(\mathbf{n}, \xi)\hat{S}_a \right.$$
$$\left. + \hat{S}_a\hat{T}^{\dagger}(\mathbf{n}, \xi)\hat{S}_a\hat{T}(\mathbf{n}, \xi))|\psi\rangle \right) \tag{18}$$

4. SU(2) COHERENT STATES

The task we now face is to establish upper bounds on the fidelities η_i, η_f (or, equivalently, lower bounds on the errors $\Delta_{ei}S$, $\Delta_{ef}S$) and then to establish the form of the operators $\hat{T}(\mathbf{n}, \xi)$, $\hat{E}(\mathbf{n})$ for which these bounds are achieved. The theory of SU(2) coherent states will play an important role in the argument. In order to fix notation, we begin by summarizing the relevant parts of this theory. For proofs of the statements made in this section see Radcliffe (1971), Arecchi *et al.* (1972), Lieb (1973), Berezin (1975), Perelomov (1986), Várilly and Gracia Bondía (1989), and Amiet and Cibils (1991).

For each unit vector $\mathbf{n} \in \mathbb{R}^3$ choose a vector $\boldsymbol{\theta}_\mathbf{n} \in \mathbb{R}^3$ with the property

$$\exp[-i\boldsymbol{\theta}_\mathbf{n} \cdot \hat{\mathbf{S}}]\hat{S}_3 \exp[i\boldsymbol{\theta}_\mathbf{n} \cdot \hat{\mathbf{S}}] = \mathbf{n} \cdot \hat{\mathbf{S}}$$

Define

$$|\mathbf{n}, m\rangle = \exp[-i\boldsymbol{\theta}_\mathbf{n} \cdot \hat{\mathbf{S}}]|m\rangle \tag{19}$$

where $|m\rangle$ is the normalized eigenvector of $\hat{S}_3$ with eigenvalue m. We then have

$$\mathbf{n} \cdot \hat{\mathbf{S}}|\mathbf{n}, m\rangle = m|\mathbf{n}, m\rangle \tag{20}$$

and

$$\frac{2s + 1}{4\pi} \int d\mathbf{n} \, |\mathbf{n}, \mathbf{m}\rangle\langle\mathbf{n}, m| = 1$$

for all m.

We are especially interested in the states $|\mathbf{n}, s\rangle$. These are the minimum-uncertainty states, for which $\Sigma_{a=1}^3 (\Delta \hat{S}_a)^2 = s$. To denote them, we employ the abbreviated notation

$$|\mathbf{n}\rangle = |\mathbf{n}, s\rangle \tag{21}$$

The states $|\mathbf{n}\rangle$ so defined $\in \mathcal{H}_{sy}$ and are eigenvectors of $\mathbf{n} \cdot \hat{S}$. They need to be carefully distinguished from the states $|\mathbf{n}, \xi\rangle$ which $\in \mathcal{H}_{ap}$ and are eigenvectors of $\hat{\mathbf{n}}$.

Let $\hat{A}$ be any operator acting on $\mathcal{H}_{sy}$. The covariant symbol corresponding to $\hat{A}$ is defined by

$$A_{cv}(\mathbf{n}) = \langle\mathbf{n}|\hat{A}|\mathbf{n}\rangle$$

The contravariant symbol corresponding to $\hat{A}$ is defined to be the unique function A_{cn} for which

$$\hat{A} = \frac{2s + 1}{4\pi} \int d\mathbf{n} \, A_{cn}(\mathbf{n})|\mathbf{n}\rangle\langle\mathbf{n}|$$

and which satisfies

$$\int d\mathbf{n} \, \Pi_{2s}(\mathbf{n}, \mathbf{n}')A_{cn}(\mathbf{n}') = A_{cn}(\mathbf{n})$$

where $\Pi_{2s}(\mathbf{n}, \mathbf{n}')$ is the projection kernel

$$\Pi_{2s}(\mathbf{n}, \mathbf{n}') = \sum_{j=0}^{2s} \sum_{m=j}^{j} Y_{jm}(\mathbf{n})Y_{jm}^*(\mathbf{n}') = \sum_{j=0}^{2s} \frac{2j + 1}{4\pi} P_j(\mathbf{n} \cdot \mathbf{n}') \tag{22}$$

In these expressions the Y_{jm} are spherical harmonics and the P_j are Legendre polynomials.

It can be shown that, given any square-integrable function f,

$$\hat{A} = \frac{2s + 1}{4\pi} \int d\mathbf{n} \, f(\mathbf{n})|\mathbf{n}\rangle\langle\mathbf{n}| \tag{23}$$

if and only if

$$\int d\mathbf{n}' \, \Pi_{2s}(\mathbf{n}, \mathbf{n}')f(\mathbf{n}') = A_{cn}(\mathbf{n}) \tag{24}$$

for almost all $\mathbf{n}$.

The covariant (respectively contravariant) symbol of an operator is often referred to as the Q (respectively P) symbol of that operator. However, we will find it more convenient to reserve this notation for the symbols corresponding specifically to the density matrix, scaled by a factor $(2s + 1)/(4\pi)$:

$$Q(\mathbf{n}) = \frac{2s + 1}{4\pi} \, \rho_{cv}(\mathbf{n}) \tag{25}$$

$$P(\mathbf{n}) = \frac{2s + 1}{4\pi} \, \rho_{cn}(\mathbf{n}) \tag{26}$$

With this rescaling the Q- and P-functions satisfy the normalization condition

$$\int d\mathbf{n} \, Q(\mathbf{n}) = \int d\mathbf{n} \, P(\mathbf{n}) = 1$$

In particular, $Q(\mathbf{n})$ is a probability density function. As we will see, it is in fact the probability density function describing the outcome of a retrodictively optimal type 1 measurement.

5. RETRODICTIVELY OPTIMAL TYPE 1 MEASUREMENTS

The purpose of this section is to investigate those processes which maximize the retrodictive fidelity. We begin by establishing the following bound on η_i:

$$\eta_i \leq s \tag{27}$$

which, in view of Eq. (13), implies

$$\Delta_{ei}S \geq \sqrt{s} \tag{28}$$

We will refer to Inequality (28) as the retrodictive error relation. It can be seen that it has the same form as the ordinary uncertainty relation, $\Delta S \geq \sqrt{s}$. It is the analogue, for the kind of measurement here considered, of the inequality $\Delta_{ei}x \, \Delta_{ei}p \geq 1/2$ proved in Appleby (1998c) for joint measurements of position and momentum (in units such that $\hbar = 1$).

In order to prove this result, we note that it follows from Eqs. (4) and (16) that

$$(2s + 1)\eta_i \leq \int d\mathbf{n} \, d\xi \, \mathrm{Tr}(\mathbf{n} \cdot \hat{\mathbf{S}}\hat{T}^\dagger(\mathbf{n}, \xi)\hat{T}(\mathbf{n}, \xi))$$

In view of Eqs. (4) and (6) we also have

$$\int d\mathbf{n} \, d\xi \, \mathrm{Tr}(\hat{T}^{\dagger}(\mathbf{n}, \xi)\hat{T}(\mathbf{n}, \xi)) = (2s + 1)$$

Consequently,

$$\int d\mathbf{n} \, d\xi \, \mathrm{Tr}((\eta_i - \mathbf{n} \cdot \hat{\mathbf{S}})\hat{T}^{\dagger}(\mathbf{n}, \xi)\hat{T}(\mathbf{n}, \xi)) \leq 0 \qquad (29)$$

For each fixed $\mathbf{n}$ the kets $|\mathbf{n}, m\rangle$ defined by Eq. (19) constitute an orthonormal basis. We may therefore write

$$\hat{T}(\mathbf{n}, \xi) = \sum_{m,m'=-s}^{s} T_{mm'}(\mathbf{n}, \xi)|\mathbf{n}, m\rangle\langle\mathbf{n}, m'| \qquad (30)$$

for suitable coefficients $T_{mm'}$. Substituting this expression in Inequality (29) gives

$$\sum_{m,m'=-s}^{s} \left((\eta_i - m') \int d\mathbf{n} \, d\xi \, |T_{mm'}(\mathbf{n}, \xi)|^2 \right) \leq 0 \qquad (31)$$

Inequality (27) is now immediate.

We next show that the retrodictive fidelity achieves its maximum value $\eta_i = s$ if and only if $\hat{E}(\mathbf{n})$ is of the form

$$\hat{E}(\mathbf{n}) = \frac{2s + 1}{4\pi} \, g(\mathbf{n})|\mathbf{n}\rangle\langle\mathbf{n}| \qquad (32)$$

for almost all $\mathbf{n}$, where $|\mathbf{n}\rangle$ is the state defined by Eq. (21), and where g is any function satisfying

$$\int d\mathbf{n}' \, \Pi_{2s}(\mathbf{n}, \mathbf{n}')g(\mathbf{n}') = 1 \qquad (33)$$

where $\Pi_{2s}(\mathbf{n}, \mathbf{n}')$ is the projection kernel defined by Eq. (22).

In fact, setting $\eta_i = s$ in Inequality (31) gives

$$\sum_{m,m'=-s}^{s} \left((s - m') \int d\mathbf{n} \, d\xi \, |T_{mm'}(\mathbf{n}, \xi)|^2 \right) \leq 0 \qquad (34)$$

from which it follows that the coefficients $T_{mm'}$ must be of the form

$$T_{mm'}(\mathbf{n}, \xi) = \left(\frac{2s + 1}{4\pi} \right)^{1/2} \delta_{m'l}g_m(\mathbf{n}, \xi)$$

for almost all $\mathbf{n}$, ξ. Substituting this expression into Eq. (30) gives

$$\hat{T}(\mathbf{n}, \xi) = \left(\frac{2s + 1}{4\pi}\right)^{1/2} |g(\mathbf{n}, \xi)\rangle\langle\mathbf{n}| \tag{35}$$

for almost all $\mathbf{n}$, ξ, where

$$|g(\mathbf{n}, \xi)\rangle = \sum_{m=-s}^{s} g_m(\mathbf{n}, \xi)|\mathbf{n}, m\rangle$$

Setting

$$g(\mathbf{n}) = \int d\xi \||g(\mathbf{n}, \xi)\rangle\|^2$$

and using Eq. (4), we deduce that $\hat{E}(\mathbf{n})$ is of the form specified by Eq. (32). It follows from Eqs. (6), (23), and (24) and the fact that $\mathrm{id}_{cn}(\mathbf{n}) = 1$ that the function g must satisfy Eq. (33). This proves that the condition represented by Eqs. (32) and (33) is necessary.

Suppose, on the other hand, that $\hat{E}(\mathbf{n})$ is given by Eq. (32), with g satisfying Eq. (33). Using Eqs. (16), (23), and (24) we deduce

$$\eta_i = \inf_{|\psi\rangle \in \mathcal{S}_{sy}} \left(\frac{2s + 1}{4\pi} \int d\mathbf{n} \, sg(\mathbf{n})|\langle\mathbf{n}|\psi\rangle|^2\right) = s$$

which shows that the condition is also sufficient.

The condition $\eta_i = s$ is not, by itself, enough to determine the distribution of measured values. However, the requirement that the retrodictive fidelity be maximized is not the only property which it is natural to require of a measurement that is to count as optimal. It is also natural to require that the measurement does not pick out any distinguished spatial directions. We accordingly define an isotropic measurement to be one which has the property that, if the initial system state density matrix takes the rotationally invariant form

$$\hat{\rho}_i = \frac{1}{2s + 1}$$

then the distribution of measured values is also rotationally invariant:

$$\rho_{val}(\mathbf{n}) = \frac{1}{4\pi}$$

for all $\mathbf{n}$.

We define a retrodictively optimal type 1 measurement process to be an isotropic process for which the retrodictive fidelity is maximal, $\eta_i = s$. It is then straightforward to verify that a type 1 measurement process is

retrodictively optimal if and only if $\hat{E}(\mathbf{n}) = (2s + 1)/(4\pi)|\mathbf{n}\rangle\langle\mathbf{n}|$. This is the POVM which has previously been discussed by Busch and Schroeck (1989), Grabowski (1989), Peres (1993), Busch *et al*, (1995), and Schroeck (1996).

We see from Eq. (8) that the measurement is retrodictively optimal if and only if the distribution of measured values is given by

$$\rho_{\text{val}}(\mathbf{n}) = Q_{\text{i}}(\mathbf{n})$$

for all $\mathbf{n}$, where Q_{i} is the Q-function corresponding to the initial system state density matrix:

$$Q_{\text{i}}(\mathbf{n}) = \frac{2s + 1}{4\pi} \langle\mathbf{n}|\hat{\rho}_{\text{i}}|\mathbf{n}\rangle$$

In terms of the operator $\hat{T}(\mathbf{n}, \xi)$, the necessary and sufficient condition for a type 1 measurement to be retrodictively optimal is that [see Eq. (35)]

$$\hat{T}(\mathbf{n}, \xi) = \left(\frac{2s + 1}{4\pi}\right)^{1/2} |g(\mathbf{n}, \xi)\rangle\langle\mathbf{n}| \tag{36}$$

where $|g(\mathbf{n}, \xi)\rangle$ is any family of kets with the property

$$\int d\xi \, \||g(\mathbf{n}, \xi)\rangle\|^2 = 1 \tag{37}$$

for all $\mathbf{n}$.

We conclude this section by showing that for retrodictively optimal type 1 measurements, $\langle\hat{\mathbf{S}}_{\text{i}}\rangle = (s + 1)\langle\hat{\mathbf{n}}_{\text{f}}\rangle$. In fact,

$$\langle\psi \otimes \chi_{\text{ap}}|\hat{\mathbf{n}}_{\text{f}}|\psi \otimes \chi_{\text{ap}}\rangle = \int d\mathbf{n} \, \mathbf{n}\langle\psi|\hat{E}(\mathbf{n})|\psi\rangle$$

$$= \frac{2s + 1}{4\pi} \int d\mathbf{n} \, \mathbf{n}|\langle\psi|\mathbf{n}\rangle|^2$$

$$= \frac{1}{s + 1} \langle\psi \otimes \chi_{\text{ap}}|\hat{\mathbf{S}}_{\text{i}}|\psi \otimes \chi_{\text{ap}}\rangle$$

where we have used the fact (Lieb, 1973) that $(s + 1)\mathbf{n}$ is the contravariant symbol corresponding to $\hat{\mathbf{S}}$.

6. PREDICTIVELY OPTIMAL TYPE 1 MEASUREMENTS

The purpose of this section is to characterize the form of the operator $\hat{T}(\mathbf{n}, \xi)$ and function $\rho_{\text{val}}(\mathbf{n})$ for processes which maximize the predictive fidelity, η_{f}. In the last section we showed that, for retrodictively optimal type

1 measurements, ρ_{val} coincides with the initial system state Q-function. In this section we will show that if the measurement is predictively optimal, then ρ_{val} is related to the final system state P-function.

We begin by establishing an upper bound on η_f. By a similar argument to the one leading to Inequality (29) we find

$$\int d\mathbf{n} \, d\xi \, \mathrm{Tr}((\eta_f - \mathbf{n} \cdot \hat{\mathbf{S}})\hat{T}(\mathbf{n}, \xi)\hat{T}^{\dagger}(\mathbf{n}, \xi)) \leq 0$$

which only differs from Inequality (29) in the replacement of η_i by η_f and in the fact that the order of $\hat{T}(\mathbf{n}, \xi)$ and $\hat{T}^{+}(\mathbf{n}, \xi)$ is reversed. The analysis therefore proceeds in nearly the same way. Corresponding to Inequality (27), we deduce

$$\eta_f \leq s \tag{38}$$

which, in view of Eq. (14), implies

$$\Delta_{\mathrm{ef}}S \geq \sqrt{s} \tag{39}$$

We will refer to Inequality (39) as the predictive error relation. It is the analogue, for measurements of spin direction, of the inequality $\Delta_{\mathrm{ef}}x \, \Delta_{\mathrm{ef}}p \geq 1/2$ proved in Appleby (1998c) for joint measurements of position and momentum (units chosen such that $\hbar = 1$).

We define a predictively optimal type 1 measurement to be one for which the predictive fidelity is maximal, $\eta_f = s$ (unlike the case of retrodictive optimality, we do not impose the requirement that the measurement also be isotropic). By a similar argument to the one given in the last section we find, corresponding to Eqs. (36) and (37), that the necessary and sufficient condition for a type 1 measurement to be predictively optimal is that $\hat{T}(\mathbf{n}, \xi)$ be of the form

$$\hat{T}(\mathbf{n}, \xi) = \left(\frac{2s + 1}{4\pi}\right)^{1/2} |\mathbf{n}\rangle\langle h(\mathbf{n}, \xi)| \tag{40}$$

for almost all $\mathbf{n}$, ξ, where $|h(\mathbf{n}, \xi)\rangle$ is any family of kets satisfying the completeness relation

$$\frac{2s + 1}{4\pi} \int d\mathbf{n} \, d\xi \, |h(\mathbf{n}, \xi)\rangle\langle h(\mathbf{n}, \xi)| = 1 \tag{41}$$

If $\hat{T}(\mathbf{n}, \xi)$ is of this form, it follows, from Eqs. (4) and (8) that

$$\rho_{val}(\mathbf{n}) = \frac{2s + 1}{4\pi} \int d\xi \, \langle h(\mathbf{n}, \xi)|\hat{\rho}_i|h(\mathbf{n}, \xi)\rangle \tag{42}$$

where $\hat{\rho}_i$ is the initial system density matrix. Now suppose that the measured

value of **n** has been recorded to lie in the region $\mathcal{R}$ of the unit 2-sphere. Then, using Eqs. (9), (40), and (42), we find

$$\hat{\rho}_f = \frac{1}{p_{\mathcal{R}}} \int_{\mathcal{R}} d\mathbf{n} \, \rho_{\text{val}}(\mathbf{n})|\mathbf{n}\rangle\langle\mathbf{n}|$$

where $p_{\mathcal{R}}$ is the probability of recording the result $\mathbf{n} \in \mathcal{R}$, and where $\hat{\rho}_f$ is the final system reduced density matrix. In view of Eqs. (23), (24), and (26) this means that the final system state P-function P_f is given by

$$P_f(\mathbf{n}) = \frac{1}{p_{\mathcal{R}}} \int_{\mathcal{R}} d\mathbf{n}' \, \Pi_{2s}(\mathbf{n}, \mathbf{n}')\rho_{\text{val}}(\mathbf{n}')$$

for almost all **n**, where Π_{2s} is the projection kernel defined by Eq. (22).

If $\mathcal{R}$ is a sufficiently small region surrounding the point $\mathbf{n}_0$, then

$$\hat{\rho}_f \approx |\mathbf{n}_0\rangle\langle\mathbf{n}_0|$$

Finally, we note that for a predictively optimal type 1 measurement, $\langle\hat{\mathbf{S}}_f\rangle = s\langle\hat{\mathbf{n}}_f\rangle$. In fact,

$$\langle\psi \otimes \chi_{\text{ap}}|\hat{\mathbf{S}}_f|\psi \otimes \chi_{\text{ap}}\rangle = \int d\mathbf{n} \, d\xi \, \langle\psi|\hat{T}^\dagger(\mathbf{n}, \xi)\hat{\mathbf{S}}\hat{T}(\mathbf{n}, \xi)|\psi\rangle$$

$$= \frac{2s + 1}{4\pi} \int d\mathbf{n} \, d\xi \, \langle\mathbf{n}|\hat{\mathbf{S}}|\mathbf{n}\rangle\langle\psi|h(\mathbf{n}, \xi)\rangle\langle h(\mathbf{n}, \xi)|\psi\rangle$$

$$= s \int d\mathbf{n} \, \mathbf{n}\langle\psi|\hat{E}(\mathbf{n})|\psi\rangle$$

$$= s\langle\psi \otimes \chi_{\text{ap}}|\hat{\mathbf{n}}_f|\psi \otimes \chi_{\text{ap}}\rangle$$

where we have used the fact (Lieb, 1973) that $s\mathbf{n}$ is the covariant symbol corresponding to $\hat{\mathbf{S}}$.

7. COMPLETELY OPTIMAL TYPE 1 MEASUREMENTS

We define a completely optimal type 1 measurement to be one which is both retrodictively and predictively optimal. Referring to Eqs. (36), (37), (40), and (41), we see that the necessary and sufficient condition for this to be true is that $\hat{T}(\mathbf{n}, \xi)$ be of the form

$$\hat{T}(\mathbf{n}, \xi) = \left(\frac{2s + 1}{4\pi}\right)^{1/2} f(\mathbf{n}, \xi)|\mathbf{n}\rangle\langle\mathbf{n}|$$

where f is any function with the property

$$\int d\xi \, |f(\mathbf{n}, \xi)|^2 = 1$$

for all $\mathbf{n}$.

Expressed in terms of the operator $\hat{U}$, the condition reads [see Eq. (1)]

$$(\langle m_1| \otimes \langle \mathbf{n}, \xi|)\hat{U}(|m_2\rangle \otimes |\chi_{\mathrm{ap}}\rangle) = \left(\frac{2s+1}{4\pi}\right)^{1/2} f(\mathbf{n}, \xi)\langle m_1|\mathbf{n}\rangle\langle \mathbf{n}|m_2\rangle$$

It is straightforward to verify that there do exist unitary operators $\hat{U}$ with this property. It follows that completely optimal measurements are defined mathematically. The question as to whether they are possible physically is, of course, rather less straightforward.

Referring to Eq. (18), we see that, for a completely optimal measurement, the quantity η_{d} characterizing the extent to which the system is disturbed by the measurement process is given by

$$\eta_{\mathrm{d}} = \inf_{|\psi\rangle \in \mathcal{S}_{\mathrm{sy}}} \left(\frac{2s+1}{4\pi} \int d\mathbf{n} \, \frac{s}{2} \, (\langle\psi|\mathbf{n}\rangle\langle\mathbf{n}|\mathbf{n} \cdot \hat{\mathbf{S}}|\psi\rangle + \langle\psi|\mathbf{n} \cdot \hat{\mathbf{S}}|\mathbf{n}\rangle\langle\mathbf{n}|\psi\rangle)\right) = s^2$$

(43)

where we have used the fact (Lieb, 1973) that $s\mathbf{n}$ is the covariant symbol corresponding to $\hat{\mathbf{S}}$. In view of Eq. (15) it follows that

$$\Delta_{\mathrm{d}}S = \sqrt{2s}$$

8. TYPE 2 MEASUREMENTS

In the preceding sections we have been concerned with type 1 measurements, for which the pointer position is constrained to lie on the unit 2-sphere. We now turn our attention to type 2 measurements. As explained in the Introduction, these are measurements for which the outcome is represented by the three *independent* commuting components of a vector $\hat{\boldsymbol{\mu}}$, no constraint being placed on the squared modulus $\hat{\mu}^2 = \Sigma_{a=1}^3 \hat{\mu}_a^2$. We will show that the more nearly a type 2 measurement approaches optimality, the more nearly it approximates an (optimal) type 1 measurement.

We first need to characterize the accuracy of a type 2 measurement. A similar analysis to that given in Section 2 can be carried through for type 2 measurements, with the replacement $\mathbf{n} \rightarrow \boldsymbol{\mu}$. As before, we denote the additional apparatus degrees of freedom $\hat{\xi} = (\hat{\xi}_1, \ldots, \hat{\xi}_N)$, so that the eigenkets $|\boldsymbol{\mu}, \xi\rangle$ comprise an orthonormal basis for the apparatus state space $\mathcal{H}_{\mathrm{ap}}$. Let $|\chi_{\mathrm{ap}}\rangle$ be the intial apparatus state and let $\hat{U}$ be the unitary operator describing the evolution brought about by the measurement interaction. Then, if the

initial system state is $|\psi\rangle$, the final state of system + apparatus, immediately after the measurement interaction has ended, will be given by $\hat{U}|\psi \otimes \chi_{\text{ap}}\rangle$. Corresponding to Eqs. (1) and (4), we define

$$\hat{T}(\boldsymbol{\mu}, \xi) = \sum_{m,m'=-s}^{s} (\langle m| \otimes \langle \boldsymbol{\mu}, \xi|)\hat{U}(|m'\rangle \otimes |\chi_{\text{ap}}\rangle)|m\rangle\langle m'|$$

and

$$\hat{E}(\boldsymbol{\mu}) = \int d\xi \; \hat{T}^{\dagger}(\boldsymbol{\mu}, \xi)\hat{T}(\boldsymbol{\mu}, \xi) \tag{44}$$

Corresponding to Eqs. (13) and (14), we define the maximal rms errors of retrodiction and prediction by

$$\Delta_{\text{ei}}S = \left(\sup_{|\psi\rangle \in \mathcal{S}_{\text{sy}}} (\langle\psi \otimes \chi_{\text{ap}}||\hat{\boldsymbol{\mu}}_{\text{f}} - \hat{\mathbf{S}}_{\text{i}}|^2|\psi \otimes \chi_{\text{ap}}\rangle) \right)^{1/2} \tag{45}$$

and

$$\Delta_{\text{ef}}S = \left(\sup_{|\psi\rangle \in \mathcal{S}_{\text{sy}}} (\langle\psi \otimes \chi_{\text{ap}}||\hat{\boldsymbol{\mu}}_{\text{f}} - \hat{\mathbf{S}}_{\text{f}}|^2|\psi \otimes \chi_{\text{ap}}\rangle) \right)^{1/2} \tag{46}$$

where $\hat{\mathbf{S}}_{\text{i}} = \hat{\mathbf{S}}$, $\hat{\mathbf{S}}_{\text{f}} = \hat{U}^{\dagger}\hat{\mathbf{S}}\hat{U}$, and $\hat{\boldsymbol{\mu}}_{\text{f}} = \hat{U}^{\dagger}\hat{\mathbf{U}}\hat{U}$. It can be seen that Eq. (45) agrees with Eq. (13) if one replaces $\hat{\boldsymbol{\mu}}_{\text{f}} \to \eta_{\text{i}}\hat{\mathbf{n}}_{\text{f}}$ and that Eq. (46) agrees with Eq. (14) if one replaces $\hat{\boldsymbol{\mu}}_{\text{f}} \to \eta_{\text{f}}\hat{\mathbf{n}}_{\text{f}}$.

In terms of the operators $\hat{E}(\boldsymbol{\mu})$ and $\hat{T}(\boldsymbol{\mu}, \xi)$ we have

$$\Delta_{\text{ei}}S = \left(\sup_{|\psi\rangle \in \mathcal{S}_{\text{sy}}} \left(\int d\boldsymbol{\mu} \sum_{a=1}^{3} \langle\psi|(\mu_a - \hat{S}_a)\hat{E}(\boldsymbol{\mu})(\mu_a - \hat{S}_a)|\psi\rangle \right) \right)^{1/2} \tag{47}$$

and

$$\Delta_{\text{ef}}S = \left(\sup_{|\psi\rangle \in \mathcal{S}_{\text{sys}}} \left(\int d\boldsymbol{\mu} \; d\xi \; \langle\psi|\hat{T}^{\dagger}(\boldsymbol{\mu}, \xi)|\hat{\boldsymbol{\mu}} - \hat{\mathbf{S}}|^2\hat{T}(\boldsymbol{\mu}, \xi)|\psi\rangle \right) \right)^{1/2} \tag{48}$$

We next show that, corresponding to Inequality (28), one has the retrodictive error relationship for type 2 measurements

$$\Delta_{\text{ef}}S \geq \sqrt{s} \tag{49}$$

and that, corresponding to Inequality (39), one has the predictive error relationship for type 2 measurements

$$\Delta_{\text{ef}}S \geq \sqrt{s} \tag{50}$$

In fact, it follows from Eqs. (44), (47), and (48) that

$$(2s + 1)(\Delta_{\mathrm{ei}}S)^2 \geq \int d\boldsymbol{\mu} \, d\xi \, \mathrm{Tr}(|\boldsymbol{\mu} - \hat{\mathbf{S}}|^2 \hat{T}^\dagger(\boldsymbol{\mu}, \xi)\hat{T}(\boldsymbol{\mu}, \xi))$$

and

$$(2s + 1)(\Delta_{\mathrm{ef}}S)^2 \geq \int d\boldsymbol{\mu} \, d\xi \, \mathrm{Tr}(|\boldsymbol{\mu} - \hat{\mathbf{S}}|^2 \hat{T}(\boldsymbol{\mu}, \xi)\hat{T}^\dagger(\boldsymbol{\mu}, \xi))$$

Using the fact that

$$(2s + 1) = \int d\boldsymbol{\mu} \, d\xi \, \mathrm{Tr}(\hat{T}^\dagger(\boldsymbol{\mu}, \xi)\hat{T}(\boldsymbol{\mu}, \xi))$$

we deduce

$$\int d\boldsymbol{\mu} \, d\xi \, \mathrm{Tr}((|\boldsymbol{\mu} - \hat{S}|^2 - (\Delta_{\mathrm{ei}}S)^2)\hat{T}^\dagger(\boldsymbol{\mu}, \xi)\hat{T}(\boldsymbol{\mu}, \xi)) \leq 0 \qquad (51)$$

and

$$\int d\boldsymbol{\mu} \, d\xi \, \mathrm{Tr}((|\boldsymbol{\mu} - \hat{S}|^2 - (\Delta_{\mathrm{ef}}S)^2)\hat{T}(\boldsymbol{\mu}, \xi)\hat{T}^\dagger(\boldsymbol{\mu}, \xi)) \leq 0 \qquad (52)$$

Now make the expansion

$$\hat{T}(\boldsymbol{\mu}, \xi) = \sum_{m,m'=-s}^{s} T_{mm'}(\boldsymbol{\mu}, \xi)|\mathbf{n}, m\rangle\langle\mathbf{n}, m'|$$

where $\mathbf{n} = \boldsymbol{\mu}/\mu$ and $|\mathbf{n}, m\rangle$ is the state defined by Eq. (19). Using this expansion, we see that Inequalities (51) and (52) become

$$\sum_{m,m'=-s}^{s} \int d\boldsymbol{\mu} \, d\xi \, ((\mu - m')^2 + (s^2 - m'^2) + (s \qquad (53)$$

$$- (\Delta_{\mathrm{ei}}S)^2))|T_{mm'}(\boldsymbol{\mu}, \xi)|^2 \leq 0$$

and

$$\sum_{m,m'=-s}^{s} \int d\boldsymbol{\mu} \, d\xi \, ((\mu - m)^2 + (s^2 - m^2) + (s \qquad (54)$$

$$- (\Delta_{\mathrm{ef}}S)^2))|T_{mm'}(\boldsymbol{\mu}, \xi)|^2 \leq 0$$

Inequalities (49) and (50) are now immediate.

Setting $\Delta_{\mathrm{ei}}S = \sqrt{s}$ in Inequality (53) gives

$$\sum_{m,m'=-s}^{s} \int d\boldsymbol{\mu} \, d\xi \, ((\mu - m')^2 + (s^2 - m'^2))|T_{mm'}(\boldsymbol{\mu}, \xi)|^2 \leq 0$$

which implies

$$|T_{mm'}(\boldsymbol{\mu}, \xi)|^2 = g_m(\mathbf{n}, \xi)\, \delta_{m's}\, \delta(\mu - s)$$

for suitable functions g_m. However, this is not possible, since the square root of the δ-function is not defined. It follows that the lower bound set by Inequality (49) is not precisely achievable. Nor is the lower bound set by Inequality (50).

It is, however, possible to approach the lower bounds set by Inequalities (49) and (50) arbitrarily closely. It can be seen that as $\Delta_{\text{ei}}S \to \sqrt{s}$ (respectively, $\Delta_{\text{ef}}S \to \sqrt{s}$), then $\hat{T}(\boldsymbol{\mu}, \xi)$ and $\hat{E}(\boldsymbol{\mu})$ become more and more strongly concentrated on the surface $\mu = s$. In other words, the measurement more and more nearly approaches a type 1 measurement of maximal retrodictive (respectively, predictive) accuracy, with pointer observable $\hat{\mathbf{n}} = \hat{\boldsymbol{\mu}}/s$.

9. CONCLUSION

There are a number of ways in which one might seek to develop the results reported in this paper.

In the first place, although we showed that $\Delta_{\text{d}}S = \sqrt{2s}$ for a completely optimal type 1 measurement, we did not derive error-disturbance relationships analogous to the inequalities $\Delta_{\text{ei}}x\, \Delta_{\text{d}}p$, $\Delta_{\text{ei}}p\, \Delta_{\text{d}}x$, $\Delta_{\text{ef}}x\, \Delta_{\text{d}}p$, $\Delta_{\text{ef}}p\, \Delta_{\text{d}}x \geq 1/2$ (in units such that $\hbar = 1$) proved in Appleby (1998c) for the case of a simultaneous measurement of position and momentum. The general principles of quantum mechanics indicate (Heisenberg 1927, 1930; Braginsky and Khalili 1992) that relationships of this kind must also hold for measurements of spin direction, at least on a qualitative level. However, it appears that the problem of giving the relationships precise, numerical expression is not entirely straightforward. The question requires further investigation.

In this paper we have considered measurements of spin direction. However, the problem of simultaneously measuring just two components of spin (Prugovečki, 1977; Schroeck, 1982; Busch, 1986, 1987, 1988; Busch and Schroeck, 1989; Martens and de Muynck, 1993; Kienzler, 1998) is also important. It would be interesting to investigate the accuracy of measurements such as this and to try to characterize the POVM (or POVMs, in the plural?) describing the outcome when the measurement is optimal.

We have seen that SU(2) coherent states play an important role in the description of optimal measurements of spin direction. Ali and Prugovečki (1977) and Appleby (1999a) have shown that ordinary, Heisenberg–Weyl coherent states play an analogous role in the description of optimal joint measurements of position and momentum. It would be interesting to see if it is generally true that every system of generalized coherent states is related in this way to joint measurements of the generators of the corresponding Lie group.

There are some important questions of principle regarding measurements of a *single* spin component (Wigner, 1952; Wheeler and Zurek, 1983; Busch and Schroeck, 1989; Busch *et al.*, 1995; Garraway and Stenholm, 1999). It would be interesting to see if the approach to the problem of defining the measurement accuracy described in this paper can be used to gain some additional insight into these questions.

Finally, it is obviously important to investigate whether optimal or near-optimal determinations of spin direction can be realized experimentally.

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